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PHYSICS NYO 7970

PROPAGATION OF STRONG BLAST IN AN ATMOSPHERE OF VARYING DENSITY

by

N. Åustern(1) and J. K. Percus(2)

August 15, 1957

institute of mathematical sciences

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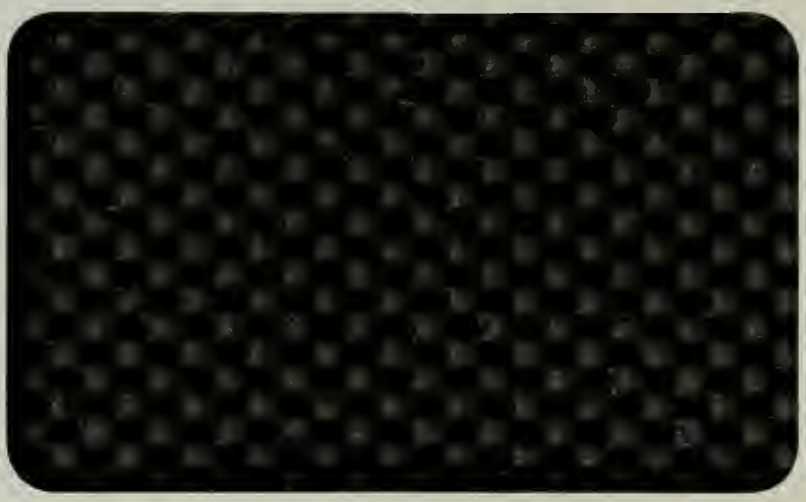

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PROPAGATION OF STRONG BLAST IN AN ATMOSPHERE OF VARYING DENSITY

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August 15, 1957

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ABSTRACT

The engulfed gas, in the propagation of strong shock through a γ-law gas when $\gamma \approx 1$, is confined to a thin shell near the shock surface. Based upon this picture, several models are constructed for the analysis of strong shock propagation in inhomogeneous media. This sequence of models utilizes successively more detailed pictures of the wave behind the shock. In order to assess the validity of the models, comparison is made with a numerical perturbation calculation of a blast in an atmosphere with linear variation of density. It is found that the distortion of the shock front and the distribution of mass per unit shock surface are predicted with considerable accuracy by the models, whereas the transverse mass flow is sensitive to the details of the model picture.

I. Introduction

A very large air blast, in the several megatons of TNT range, spreads throughout such large volumes of the atmosphere that gravity and the decrease of density with altitude might be able to influence the blast to an appreciable degree. Exactly what this influence may be can only be learned by a careful numerical integration of the equations of motion. It is desirable, nevertheless, to see what can be learned about the problem by several more qualitative methods, which give somewhat more insight into the problem.

The present report is a brief presentation of a few of the lines of approach studied by the authors.

During certain times in the motion of a real blast it is not too unlike the following model, which we study:
(1) The blast starts at a point, at the origin of coordinates,
(2) The blast is "strong" (in the sense of G. I. Taylor),
(3) The atmospheric density (which alone is of consequence for strong shock) decreases linearly with altitude. So long as the shock is strong the hydrodynamical accelerations of the blast are very much larger than the acceleration due to gravity, and g may be neglected in the differential equations of motion. Only this case is treated in the present report.

Two chief lines of approach are discussed. In one the hydrodynamical equations are replaced by simpler equations which deal only with integrated flux quantities, according

to the method of the excellent "snowplow" approximation. This approach gives a great deal of insight into the motion, and very likely can be extended into the region of weak shock. Unfortunately it is difficult to be confident of the correctness of all the details of the snowplow approach for a two dimensional problem, so some kind of standard of comparison is provided by the other approach, a numerical calculation of the blast motion by the method of perturbation theory[1]. The perturbation calculation is described first.

(1) See also
Hellund, Emil, Perturbation Treatment of Blast in Inhomogeneous Media, Argonne National Laboratory (Unpublished Report).

II. The Perturbation Calculation

Except for very large blast radii the gravity effect may be treated as a small perturbation. An analysis on this basis is described below, the unperturbed solution being taken to be the Taylor result for strong shock.

As we have noted, so long as the shock is strong the hydrodynamical accelerations of the blast are very much larger than the acceleration of gravity, and g may be neglected in the differential equations of motion. Gravity thus influences the blast, for strong shock, only in having produced the atmospheric inhomogeneity which the blast must encounter. In terms of ρ_1, the density just outside the shock, and $\vec{U}$ the shock velocity, the strong shock boundary conditions give for density, pressure, and velocity just inside the shock surface the results

$$\rho = \rho_1 (\gamma+1)/(\gamma-1) \qquad (1)$$

$$p = 2\rho_1 U^2/(\gamma+1) \qquad (2)$$

$$\vec{u} = 2\vec{U}/(\gamma+1) \qquad (3)$$

The density ρ_1 which enters these equations is approximately

$$\rho_1 = \rho_o[1 - (r/\lambda)\cos\theta] \qquad (4)$$

where the scale distance λ is about 8 kilometers, this being its value if the atmosphere is isothermal and at ordinary temperature; θ is the angle with respect to the

vertical, r the distance from blast origin. It is the second term of (4) which is the forcing term in the perturbation calculation, producing all the departures from spherical symmetry. It may be shown that the gravity effects can be relied upon to be small, the assumption of the present calculation, only for blast radii very much less than eight kilometers.

In spherical coordinates the differential equations of hydrodynamics take the form

$$\frac{\partial u_r}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}[u_r^2 + u_\theta^2] - \frac{u_\theta}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (5)$$

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\frac{\partial (ru_\theta)}{\partial r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad (6)$$

$$\left[\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right]\rho = -\rho\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, u_\theta)\right] \qquad (7)$$

$$\left[\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right]p = -\gamma p\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, u_\theta)\right]. \qquad (8)$$

These are now to be solved approximately, subject to the conditions (1) - (4), and to conditions at r = 0 which will be described later. For the spherically symmetrical problem, about which we perturb, Taylor[2] found the similarity solution:

(2) Taylor, G. I., The Formation of a Blast Wave by an Intense Explosion, RC-210 (Civil Defense Research Committee of Ministry of Home Security).

$$p = \gamma^{-1} \rho_o A^2 R_o^{-3} f(y) \tag{9}$$

$$\rho = \rho_o \, g(y) \tag{10}$$

$$u = A \, R_o^{-3/2} \, h(y) \tag{11}$$

$$y = (r/R_o) \quad , \quad R_o = (\tfrac{5}{2} At)^{2/5} \tag{12}$$

In order to establish a correspondence with the Taylor solution we now introduce for the present problem the similarity variable

$$y = (r/R) \quad , \tag{13}$$

where R is the radial distance to the shock front, as a function of θ. With the introduction of this variable the description of the problem has been transferred from the coordinates (r, θ, t) to the coordinates (y, θ, t).

As a next step the differential equations are simplified to the case of first order perturbation theory. The procedure is straightforward but lengthy. Each hydrodynamical quantity is written as an expansion in the parameter λ and substituted into the equations of motion. The zero order solution is of course given by the Taylor blast wave, while the higher order equations are all linear inhomogeneous. Separation of variables can be achieved in the first order part, separation being possible if the perturbation functions are given the form $P_n(\cos \theta) \, t^m \, f(y)$, where P_n is the n^{th} Legendre polynomial and f a function to be determined. In the

present discussion a detailed study will be given only for the solution which varies according to $t^{2/5}$ and according to the Legendre function of first order. This solution will be discussed in full.(3)

Written out explicitly, our solutions for pressure, density, material velocity components, and shock radius, have the following form

$$p = \gamma^{-1} \rho_o A^2 R_o^{-3} \left[f + \frac{R_o}{\lambda} \cos\theta\, F(y)\right] \tag{14}$$

$$\rho = \rho_o \left[g + \frac{R_o}{\lambda} \cos\theta\, G(y)\right] \tag{15}$$

$$u_r = A R_o^{-3/2} \left[h + \frac{R_o}{\lambda} \cos\theta\, H_1(y)\right] \tag{16}$$

$$u_\theta = A R_o^{-3/2} \left[\frac{R_o}{\lambda} \sin\theta\, H_2(y)\right] \tag{17}$$

$$R = R_o \left[1 + \frac{R_o}{\lambda} \cos\theta \cdot S\right] . \tag{18}$$

The functions F, G, H_1, H_2 are determined by substitution in (5) - (8) and now satisfy ordinary differential equations in the variable y. It should be noted in the above that

(3) The plausibility of the restriction of the study to the particular solution just mentioned may be indicated in the following way: Imagine a spherically symmetrical blast moving outwards from the origin at time $t = 0$. Very near the origin the second term of equation (4) is negligible, and the blast remains spherically symmetric. The spherical symmetry disappears as the motion proceeds, however, as terms proportional to the second term of (4) gradually appear in the functions which describe the blast. That these perturbation terms are proportional to the second term of (4) follows because we are working with linearized equations. Finally, the solution described in the preceding paragraph is precisely that one which has the property of being linear in the forcing term. We surmise that the remaining mathematically admissible solutions have physically inadmissible singularities, but shall not investigate the problem at this time.

$\frac{R_o}{\lambda}$ is the factor which varies as $t^{2/5}$. It also should be noted that the shock distortion is measured by the constant S. One of our principal tasks below will be the determination of S.

The conditions which remain to be satisfied may now be listed. First, substitution of (14)-(18) into (1)-(4) gives the boundary conditions at the shock surface (y = 1):

$$H_1(y=1) = 4\ S/(\gamma+1) \tag{19}$$

$$H_2(y=1) = 2\ S/(\gamma+1) \tag{20}$$

$$F(y=1) = \left(\frac{2\gamma}{\gamma+1}\right)\ [4\ S-1] \tag{21}$$

$$G(y=1) = -\ (\gamma+1)/(\gamma-1) \tag{22}$$

The differential equations themselves are most conveniently quoted in terms of the set of variables

$$x_1 = H_1, \quad x_2 = H_2, \quad x_3 = F/g, \quad x_4 = G/g, \tag{23}$$

and have the form

$$\frac{dx_i}{dy} = \sum_j A_{ij}\ x_j + SB_i\ . \tag{24}$$

The coefficients A_{ij} and B_i are complicated functions of y, involving the Taylor blast functions f, g, h and their derivatives. A tabulation of the coefficients is given in the Appendix. In the present work a solution of the equations (24) was obtained by the very straightforward procedure of step-by-step numerical integration, the integration

proceeding inwards from $y = 1$, where the boundary values (19)-(22) are known. This integration was carried through for the case $\gamma = 1.4$, and for several values of the parameter S, a parameter whose meaning must now be explained.

The parameter S first was introduced in equation (18), and entered the differential equations through the similarity transformation (13). It measures the distortion of shape of the shock surface. In some fashion S must be determined by a requirement that the solutions be consistent at $y = 0$. The requirement we actually use is that S be chosen so that the solution is "as regular as possible" at $y = 0$.

Physically, the reasonableness of determining the shock shape by a condition at the center may be understood in the following way: The hydrodynamical equations are local equations, so an approximate integration which proceeds inwards from a small region of the shock surface can only determine the properties of the volume behind the shock with respect to the properties of that surface region from which the integration has started. To determine the overall shock shape and velocity the various different surface elements must communicate with each other, by way of signals transmitted through the interior. This "communication" is equivalent mathematically to the requirement of the consistency with each other of the solutions which emanate from the different points of the shock surface. In the present problem, in which a simplified angular behavior of the solution functions

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Please note that on page 12, line 9 the word important should read unimportant.

is used (equations (14)-(18)), the consistency requirement reduces to the requirement that there be a point (or a region) near the origin where the angle dependent terms have the value zero. With only one adjustable parameter such a precise consistency requirement cannot be fulfilled exactly, and we must develop a procedure for satisfying it as well as possible. In particular, since the density becomes extremely small in the deep interior, wide fluctuations of velocity in this region will be physically important compared to density and pressure fluctuations.

In the region near $y = 0$ (a region which extends to y values as large as 0.8!) the equations (24) pass over to the form

$$\frac{dx_i}{dy} = \sum_j \bar{A}_{ij} x_j + S \bar{B}_i \quad , \tag{25}$$

where

$$\bar{A}_{ij} = \lim_{y \to 0} A_{ij} \tag{26}$$

$$\bar{B}_i = \lim_{y \to 0} B_i \tag{27}$$

These simplified coefficients are given in the Appendix. It is very easy to find all possible linearly independent solutions of equations (25). Then any one solution of equations (19)-(24) which we may be considering must go over at small y into some definite linear combination of the solutions of (25), and its behavior near $y = 0$ may be understood in terms of the properties of these elementary solutions.

The elementary solutions of (25) have the form

$$x_1 = 0.704\ Sy + \sum_{i=1}^{4} a_{1i}\ y^{\beta_i} \tag{28a}$$

$$x_2 = 0.014\ Sy + \sum a_{2i}\ y^{\beta_i} \tag{28b}$$

$$x_3 = 1.783\ Sy^2 + \sum a_{3i}\ y^{1+\beta_i} \tag{28c}$$

$$x_4 = 7.57\ S + \sum a_{4i}\ y^{-1+\beta_i} \tag{28d}$$

with

$$\left.\begin{aligned} \beta_1 &= 5.084 \\ \beta_{2,3} &= -0.645 \pm i\ 1.92 \\ \beta_4 &= -\ 9.04 \end{aligned}\right\} \tag{29}$$

Evidently the solutions involving β_2, β_3, β_4 all become infinite as $y \to 0$. Strong infinities in the functions x_3 and x_4 are especially unacceptable, as gx_3 and gx_4 are the pressure and density perturbations. On the other hand, g varies as $y^{7.5}$, so only the solution involving β_4 really corresponds to physically impossible pressure and density (so energy) distributions within the blast. We conclude that such a value of S must be chosen so that as the corresponding solutions of (24) are carried into the region $y \approx 0$ they pass over to the forms (28) with $a_{j4} = 0$.

In detail, the requirement that the β_4 solution should not occur in the numerical solution is met by forcing a

certain quantity Q to vanish. This Q is the scalar product between the adjoint of the characteristic solution corresponding to β_4, and the difference between the x_i and the particular solution of (25).

$$Q = x_1 + 0.331\, x_2 + 1.539\, \frac{x_3}{y} - 0.203(yx_4) - 1.915\, yS \quad . \quad (30)$$

We find $Q = 0$ is obtained if $S = 0.162$. It should be noted that (30) (as well as the differential equations and boundary conditions) are linear in S, so that the correct value of S may be interpolated from any two values of S and the corresponding Q's. As a numerical check, three values of S were employed in the computation.

Some interpretation of the result $S = 0.162$ is possible by brief examination of equation (18). It becomes

$$R = R_o[1 + 0.162(\frac{R_o}{\lambda}) \cos \theta] \quad , \quad (31)$$

or, the perturbation of the shock shape is only 16% of the perturbation of the ambient density. The shock remains more nearly spherical and at constant altitude than one might at first suppose. Presumably this result is a consequence of the very rapid propagation of signals across the interior of a Taylor blast, the speed of sound becoming strongly infinite as $y \dashrightarrow 0$, so that the different surface elements of the shock very easily influence each other. This same observation makes it less surprising that our approximate solution gives velocities, x_1 and x_2, which become infinite

as $y \to 0$.

The results of the perturbation calculation are tabulated in Table I, and plotted in Figures 1, 2, 3, for the value $S = .16$. In the range presented ($y = .6 - 1.0$), these results are not significantly different from those at the more accurate value of $S = .162$.

y	h(y) (=U:GIT)	$H_1(y)$	$H_2(y)$	$H_T(y)$ $=H_2-Sh$	g(y) (=ρ:GIT)	G(y)	f(y) (=p:GIT)	F(y)	S
1.00	.833	.2667	.1333	.0000	6.000	-6.000	1.166	-.4200	.16
.98	.799	.2730	.0948	-.0330	4.066	-3.435	.955	-.2732	.16
.96	.766	.2798	.0596	-.0628	2.885	-2.084	.813	-.1890	.16
.94	.737	.2866	.0289	-.0890	2.120	-1.365	.714	-.1395	.16
.92	.710	.2935	.0012	-.1123	1.600	- .950	.644	-.1078	.16
.90	.685	.3004	-.0242	-.1337	1.232	- .695	.592	-.0866	.16
.88	.662	.3071	-.0479	-.1538	.964	- .530	.554	-.0714	.16
.86	.641	.3137	-.0704	-.1729	.762	- .418	.525	-.0600	.16
.84	.621	.3199	-.0921	-.1914	.608	- .341	.503	-.0513	.16
.82	.602	.3256	-.1134	-.2099	.488	- .284	.486	-.0443	.16
.80	.584	.3307	-.1344	-.2278	.392	- .241	.472	-.0385	.16
.78	.567	.3352	-.1554	-.2461	.316	- .207	.462	-.0336	.16
.76	.550	.3389	-.1767	-.2647	.255	- .179	.454	-.0295	.16
.74	.534	.3418	-.1985	-.2839	.206	- .156	.448	-.0261	.16
.72	.519	.3437	-.2211	-.3041	.165	- .136	.442	-.0230	.16
.70	.503	.3445	-.2448	-.3252	.132	- .118	.438	-.0203	.16
.68	.488	.3440	-.2700	-.3480	.106	- .103	.435	-.0182	.16
.66	.474	.3420	-.2971	-.3729	.084	- .088	.434	-.0162	.16
.64	.459	.3382	-.3271	-.4005	.066	- .075	.431	-.0144	.16
.62	.444	.3323	-.3607	-.4317	.052	- .064	.431	-.0131	.16
.60	.430	.3238	-.3993	-.4681	.041	- .054	.430	-.0120	.16

Table 1: Numerical Values of Deviations from Taylor Flow for S = .16

Fig. 1

Amplitude of Velocity Perturbation
with Taylor Normalization

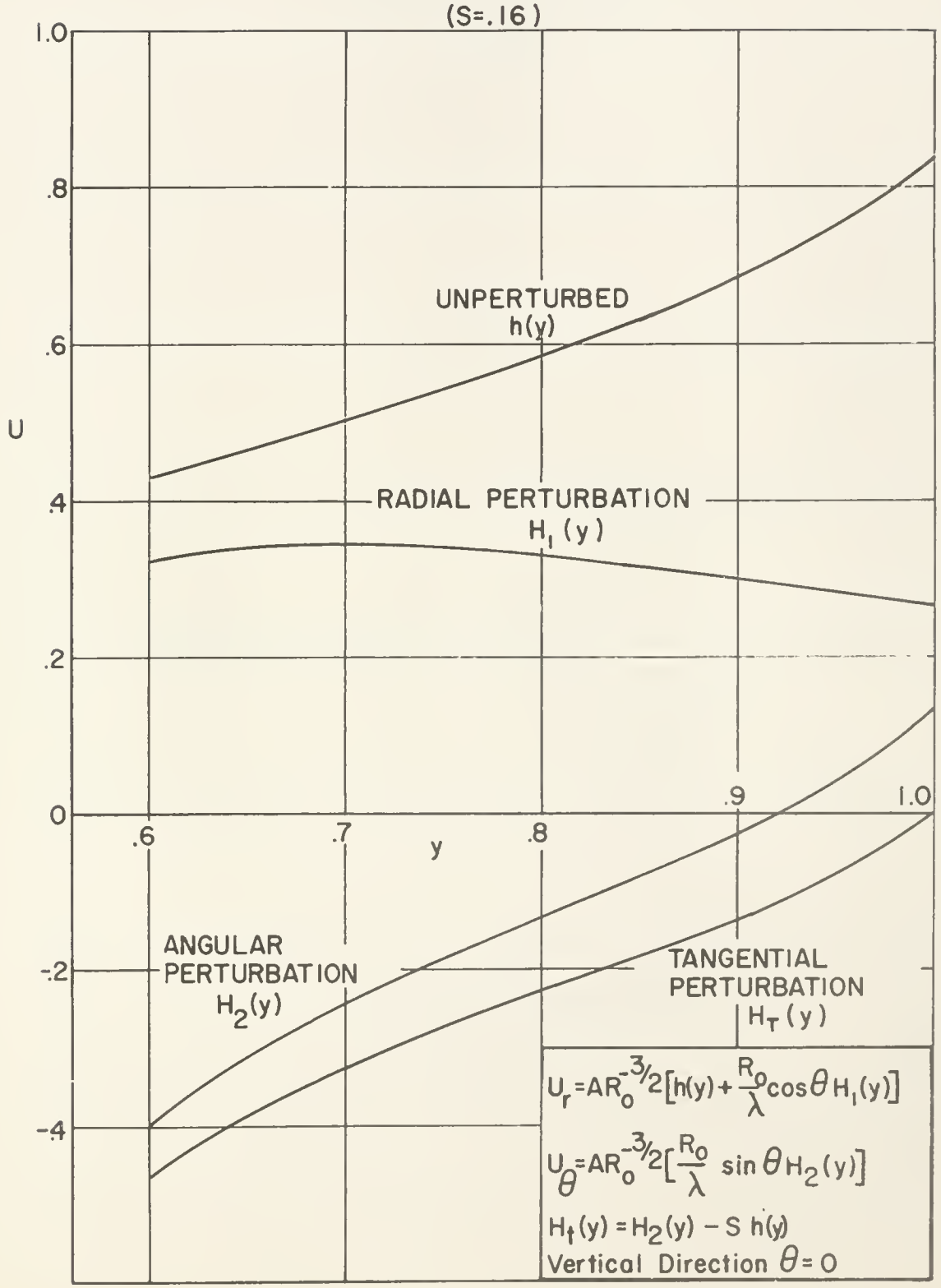


Fig. 2

Amplitude of Density Perturbation
with Taylor Normalization
(S = .16)

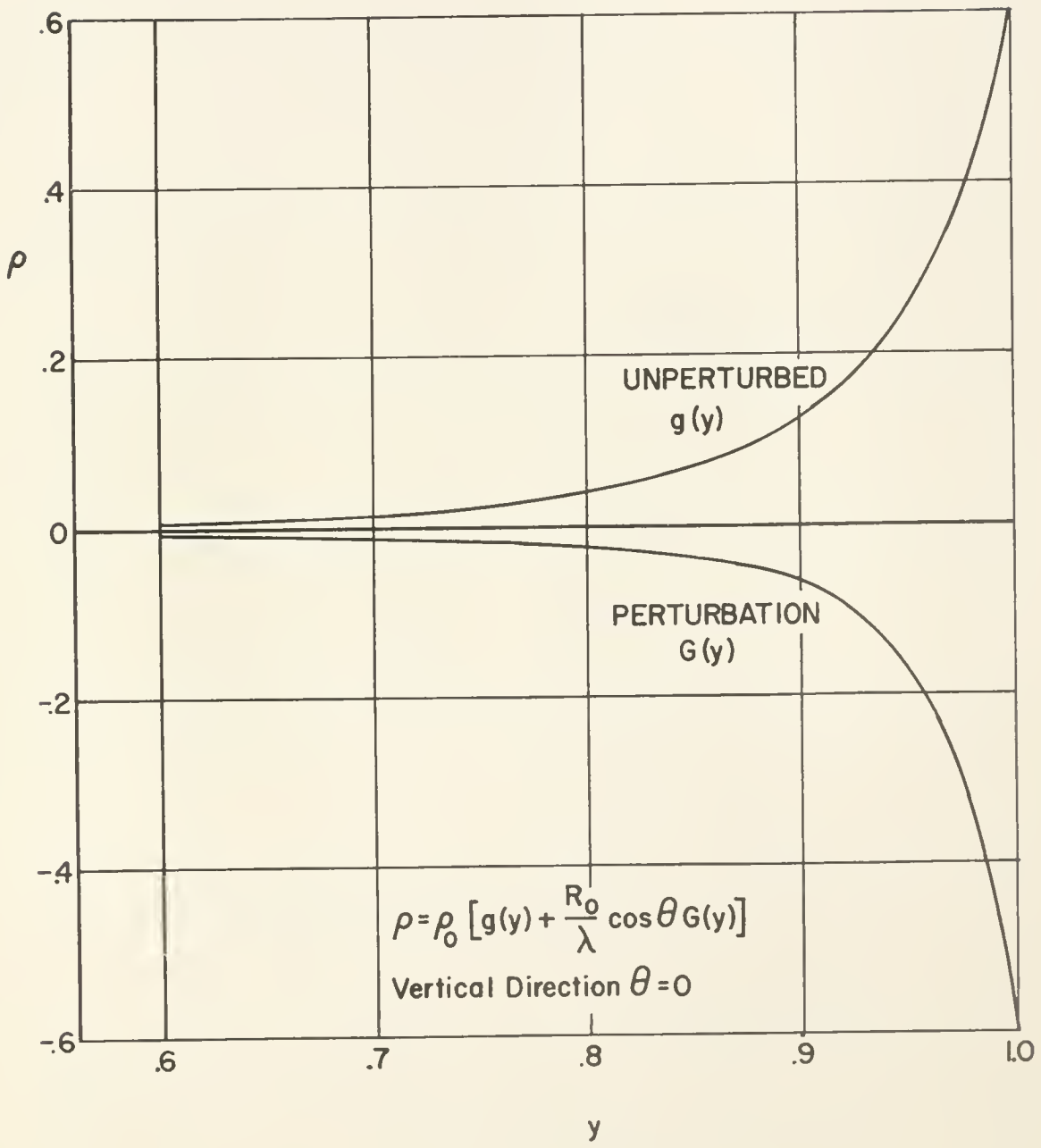
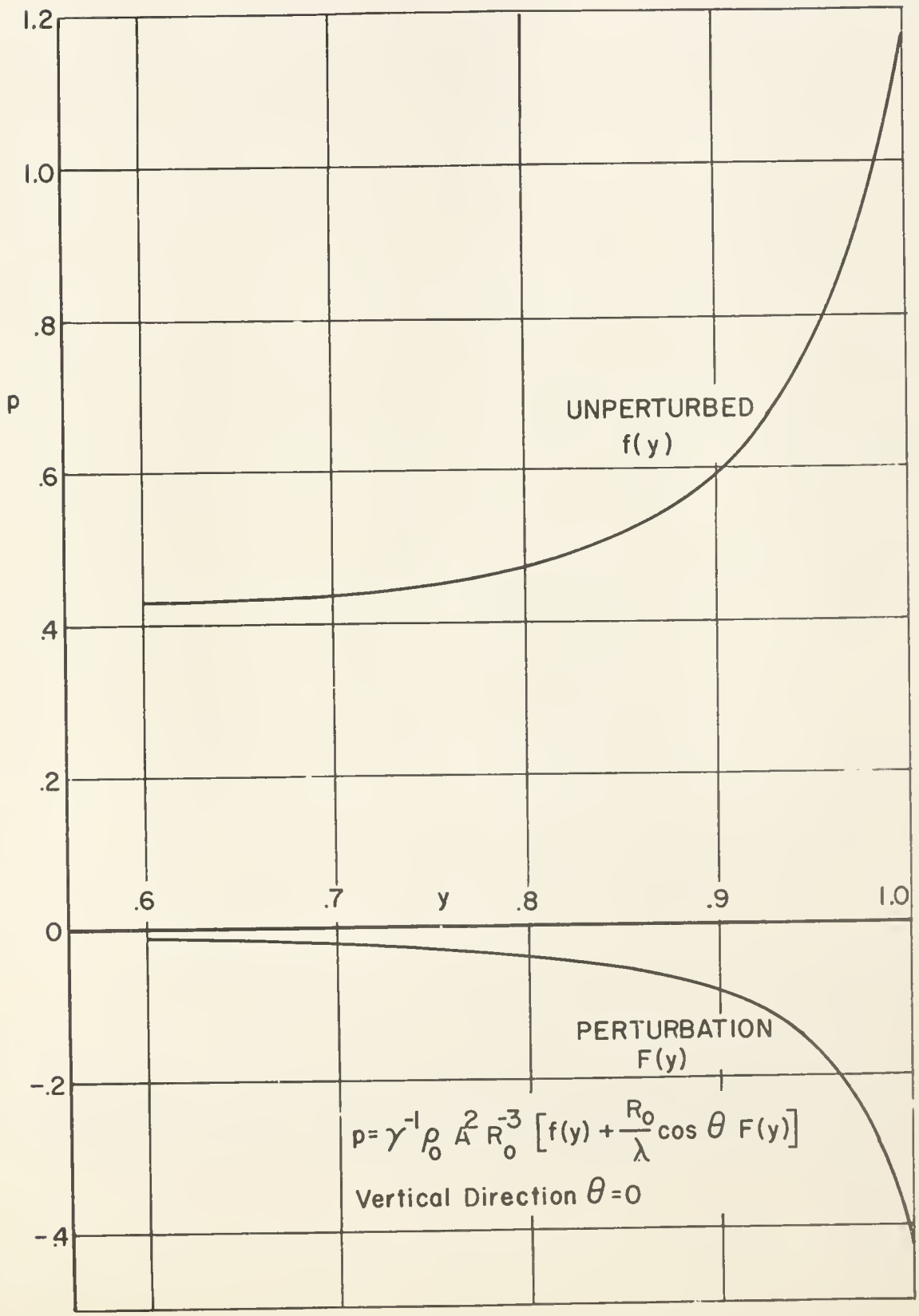


Fig. 3

Amplitude of Pressure Perturbation
with Taylor Normalization
(S=.16)



III. The Snowplow Model

A. Preliminary Results

The rather cumbersome computations of the perturbation method yield a detailed description of the hydrodynamical system under consideration. It seems reasonable to seek also a method which, while yielding less detail, may be significantly less tedious to apply and more physically direct; in brief, to calculate for a model of the system. Hints as to such models may be obtained both from our detailed numerical results and from general physical pictures. Thus, it has been observed[4] in numerical computations on inhomogeneous air blasts that the shock surface pressure, at a given instant of time, tends to be very nearly uniform under a wide variety of conditions. Since, crudely, surface pressure gradients produce mass flows which tend to cancel the gradients, such a result is not strange. If we take this picture literally, then the pressure correction F of (14) will vanish at the boundary, so that, according to (21), we have S= .25. Thus the order of magnitude of the surface distortion is correct, but such a picture is not really adequate for the strong shock region.

We now consider at some length a model whose justification is not hard to assess, and which therefore can be successively corrected with corresponding improvement

(4) Unpublished work done at New York University, 1956 by H. B. Keller, G. L. Lewis, D. L. Ludwig.

in the results obtained. It is based upon the following considerations. In a strong shock, as $\gamma \to 1$, the available compression ratio increases, so that most of the gas swept up into the shock simply gathers at the shock surface, and travels along with it. Alternatively, from (3), as $\gamma \to 1$, the surface gas velocity u approaches the shock velocity U. The net effect is that as a point blast propagates into a medium for which $\gamma \approx 1$, the engulfed gas does not spread into the interior region; it is merely "snowplowed" into a thin layer behind the shock surface. The effective removal of one dimension from the hydrodynamics introduces obvious simplifications in the analysis of the system.

Before entering into the analysis proper, let us ask what we expect to obtain from such a model; clearly, details of the wave behind the blast are washed out in this picture. The gross properties we seek will then be the surface distortion factor S (from which all surface hydrodynamical quantities are obtainable), the total mass M per unit surface, and as one aspect of the hydrodynamical structure behind the blast, the total transverse mass flow per unit surface. To compare with the perturbation calculation, we must specify the interior extension of a unit surface:

it is to be a cone, normal to the shock surface.[6]

To compute the pertinent total quantities on the basis of the perturbation solution, we observe that since the shock surface is a displaced sphere given by (18), the center line of the cone runs from the displaced sphere center at $R_o^2 \frac{S}{\lambda} \hat{z}$ to the surface point $\vec{R}$, and is of length R_o ($\hat{z}$ denotes the unit vector in the z direction). Thus if Y represents relative distance from the center of curvature, we have for any $F(\vec{r})$,

$$F_{TOT} = R_o \int_o^1 Y^2 F(Y\vec{R} + (1-Y) R_o^2 \frac{S}{\lambda} \hat{z}) \, dY \qquad (32)$$

or if

$$F(Y\vec{R}) = F(Y) + \frac{R}{\lambda} \cos\theta \, \delta F(Y) \quad , \qquad (33)$$

$$F_{TOT} = R(1-S \frac{R}{\lambda} \cos\theta) \int_o^1 Y^2 (F(Y) + \frac{R}{\lambda} \cos\theta \, \delta F(Y)) \, dY$$

$$+ \frac{R^2 S}{\lambda} \cos\theta \int_o^1 Y^2 (1-Y) F'(Y) \, dY \qquad (34)$$

which yields, after an integration by parts,

$$F_{TOT} = R[\int_o^1 Y^2 F(Y) \, dY + \frac{R}{\lambda} \cos\theta \int_o^1 Y^2 \, \delta F(Y) \, dY$$

$$- 2S \frac{R}{\lambda} \cos\theta \int_o^1 Y(1-Y) F(Y) \, dY] \quad . \qquad (35)$$

(6) Since the mass density rapidly falls to zero, the fact that the evolute is not generally a point need not concern us; in the special distortion we consider, the shock surface is a displaced sphere of radius R_o, and so the evolute actually turns out to be a point.

In particular, inserting the expression (10) for ρ, the surface mass density

$$M(R,\theta) = \rho_o R \int_o^1 [g(Y) Y^2 - 2 \frac{R}{\lambda} S \cos \theta (1-Y) Y g(Y) + \frac{R}{\lambda} \cos \theta G(Y) Y^2] dY \quad , \tag{36}$$

which, after numerical integration, results in

$$M = \frac{1}{3} \rho_o R (1 - .775 \frac{R}{\lambda} \cos \theta) \quad . \tag{37}$$

Finally, inserting the expression for the tangential velocity u_T (R,)), the mean transverse velocity becomes, to lowest order,

$$u_T = \frac{(\rho u_T)_{TOT}}{M} = \frac{AR^{-1/2}}{\lambda} \sin \theta \int_o^1 g(Y) Y^2 [H_2(Y) - Sh(Y)] dY / / \int_o^1 g(Y) Y^2 dY \quad , \tag{38}$$

from which we obtain

$$u_T = - .084 A \frac{R^{-1/2}}{\lambda} \sin \theta \quad . \tag{39}$$

We thus find that in the analytical expressions

$$R = R_o [1 + S \frac{R_o}{\lambda} \cos \theta] \tag{40a}$$

$$M = \frac{1}{3} \rho_o R [1 - m \frac{R}{\lambda} \cos \theta] \tag{40b}$$

$$u_T = - bA R^{-1/2}/\lambda \sin) \tag{40c}$$

the perturbation calculation gives for the parameters S, m, b

$$S = .162, \quad m = .775, \quad b = .084 \tag{40d}$$

In the succeeding analysis, several simple models for the hydrodynamics will be set up, and it will be seen in what manner the values these models give compare with those of (40d).

B. The Ideal Snowplow

We first analyze the case in which $\gamma \to 1$; then the mass shell has zero thickness and so the structure behind the blast may really be neglected. In order to determine the parameters S and m, we employ the integral principles of mass conservation and conservation of longitudinal momentum, together with the fact that the interior pressure is a constant, independent of θ (no mass is present to maintain a pressure gradient). Consider then a unit of shock surface, containing mass M; as it propagates with velocity $\vec{U}$ it absorbs mass $\rho_1 U$ per unit time. On the other hand, the unit surface expands on propagation, the particle direction being that of the surface normal. If the surface area is given by $\mathcal{A}$, then clearly

$$\nabla \cdot \hat{U} = [\mathcal{A}(t+dt) - \mathcal{A}(t)/\mathcal{A}(t)\, U\, dt, \qquad (41)$$

so that with $\mathcal{A}(t) = 1$, $\hat{U}$ denoting the unit vector in direction $\vec{U}$ we have

$$d\mathcal{A}/dt = U \nabla \cdot \hat{U} \ . \qquad (42)$$

The mass conservation law is thus

$$\dot{M} = U \rho_1 - M U \nabla \cdot \hat{U} \ . \qquad (43)$$

Since we have agreed to allocate all spatial quantities to the shock surface, and thus moving with it, we have quite generally

$$\dot{F} = \vec{U} \cdot \nabla F \quad . \tag{44}$$

Employing (44), (43) may now be rewritten as

$$\nabla \cdot (M \hat{U}) = \rho_1 \quad . \tag{45}$$

It is in equation (45) that the ambient density enters the calculations, leading to the non-spherical results which will be found.

Next we utilize the longitudinal component of the law of force $f = ma$ for a unit surface. Both the surface pressure $p = \rho_1 U^2$ and internal pressure, say p_i, are applied longitudinally i.e. in the direction of the surface normal $\hat{U}$. Furthermore we have

$$\hat{U} \cdot \frac{d}{dt} \vec{U} = \hat{U} \cdot (\vec{U} \cdot \nabla) \vec{U} = \frac{1}{U} \vec{U} \cdot (\vec{U} \cdot \nabla) \vec{U} \tag{46}$$
$$= \frac{1}{U} (\vec{U} \cdot \nabla) \frac{1}{2} \vec{U} \cdot \vec{U} = \vec{U} \cdot \nabla U \quad ,$$

whence

$$M \vec{U} \cdot \nabla U = p_i - \rho_1 U^2 . \tag{47}$$

Alternatively, eliminating ρ between (45) and (47), there results

$$\nabla \cdot M \vec{U} = p_i / U \tag{48}$$

which may be interpreted as the force law for the material in the elementary cone, where now the front surface of the elementary cone is taken to be on the ambiant side of the

shock.

Equations (45) and (48) are sufficient to determine m and S in (40). We must first express all pertinent surface quantities in terms of S. Since

$$R_o = (\tfrac{5}{2} At)^{2/5} \tag{12}$$

$$R = R_o \left(1 + S \frac{R_o}{\lambda} \cos\theta\right) \tag{18}$$

we have, retaining only first order terms in S,

$$t = \frac{2}{5A} R^{5/2} \left(1 - \frac{5}{2} S \frac{R}{\lambda} \cos\theta\right) \tag{49}$$

whence $\vec{U} = \nabla t / |\nabla t|^2$ or

$$\vec{U} = A\left[R^{-3/2} \left(1 + \frac{7}{2} S \frac{R}{\lambda} \cos\theta\right)\hat{r} + R^{-1/2} \frac{S}{\lambda} \sin\theta\, \hat{\theta}\right] \tag{51a}$$

$$U = AR^{-3/2} \left(1 + \frac{7}{2} S \frac{R}{\lambda} \cos\theta\right) \tag{51b}$$

$$\hat{U} = \hat{r} + S \frac{R}{\lambda} \sin\theta\, \hat{\theta} \tag{51c}$$

at the surface position (R,θ). Inserting (51) as well as the analytic expressions (40a), (40b) for M and ρ_1 into (45) and (48) we find

$$\rho_o \left(1 - \frac{R}{\lambda} \cos\theta\right) = \frac{\rho_o}{3} \left(3 - 4m \frac{R}{\lambda} \cos\theta\right) + \frac{2}{3} \rho_o S \frac{R}{\lambda} \cos\theta \tag{52}$$

$$\frac{A^2}{2} R^{-3} \rho_o \left(1 + \left[\frac{7}{2} S + \frac{4}{3} S + \frac{5}{3} \left(\frac{7}{2} S - m\right)\right] \frac{R}{\lambda} \cos\theta\right) =$$

$$= p_1(t) \ . \tag{53}$$

But p_1, a function of t alone, must then be $\propto t^{-6/5}$, or

$$p_i = \frac{A^2}{2} R^{-3} \rho_o (1 - 3S \frac{R}{\lambda} \cos \theta) \qquad [= \frac{1}{2} \rho_o U^2], \quad (54)$$

from which we obtain

$$4m - 2S = 3 \qquad 5m - 23S = 0 \qquad (55a)$$

$$S = \frac{15}{82} = .183 \qquad m = \frac{69}{82} = .842 \ . \qquad (55b)$$

Even this crude picture thus yields results tolerably close to the numerical expression (40d).

C. Corrections to Longitudinal Flow

If γ is greater than unity, the mass shell is no longer of zero thickness, and so some aspects of the structure behind the blast make their presence felt even in our integral type of formulation. These corrections are necessary even for propagation in a uniform medium (witness the exceptionally high value of $p_i/\rho_o U^2$ in (54)) which is an advantage, for explicit effects of inhomogeneity may be omitted in their discussion. Briefly, the thickness of the mass shell manifests itself in that the effective inertial mass is no longer precisely the total mass, and the active force is no longer just the difference between interior and surface pressure. There are also factors to be inserted in the relations between particle and shock velocity, and surface pressure and $\rho_1 U^2$, but these are obvious, and indeed their insertion would not have affected the results of the previous section.

Consider first the mass correction. Our statements on mass conservation hold unchanged, so that (45) remains correct (at least for the homogeneous case). Now since the mass is to be taken as moving at the surface particle velocity, we may define the effective longitudinal inertial mass M_L per unit surface by

$$\int \rho(\vec{r})\, \dot{u}_L(\vec{r})\, d^3r = M_L\, \vec{U}\cdot\nabla\, u(R,\theta) \quad , \tag{56}$$

where $\dot{u}_L$ is the longitudinal component of the material

time derivative of the particle velocity, and the integration is over the cone of unit shock surface. If we specialize to the homogeneous Taylor solution, this readily yields

$$a_1 \equiv \frac{M_L}{M} = (\gamma + 1) \int_0^1 [\tfrac{3}{2} h(Y) + (Y - h(Y)h'(Y)] \, g(Y) Y^2 \, dY, \quad (57)$$

a constant.

Next, the longitudinal force on the cone would be $p_i - p(R,\theta)$ if the pressure were to drop to p_i at once on passing inside the shock surface. Instead the quantity $p - p_i$ is rapidly but not immediately damped out, so that presumably some multiple of $p_i - p(R,\theta)$ should be subtracted from $p(R,\theta) - p_i$ itself to produce the total longitudinal force F_L; we write

$$F_L = a_2 \, (p_i - p(R,\theta)) \quad . \quad (58)$$

Since the total force is obtained by a surface integration of the pressure over the cone, we have

$$a_2 = - \int p(\vec{r}) \, \hat{U} \cdot d\vec{s} / (p_i - p(R,\theta)) \quad , \quad (59)$$

or specializing again to the homogeneous Taylor case,

$$a_2 = [f(1) - 2 \int_0^1 Y \, f(Y) \, dY] / (f(1) - f(0)) \quad , \quad (60)$$

again a constant.

Now, assuming the effect of inhomogeneity on the values of a_1 and a_2, being a correction on a correction, will not be important, our conservation and force equations

(45) and (47) become instead

$$\nabla \cdot (M \hat{U}) = \rho \tag{61}$$

$$\frac{2}{\gamma+1} a_1 M \vec{U} \cdot \nabla U = a_2 \left(p_1 - \frac{2}{\gamma+1} \rho_1 U^2\right) . \tag{62}$$

Solving as before, we find

$$p_1/p(R_o) = 1 - \frac{1}{2} \frac{a_1}{a_2} \qquad \text{and} \tag{63a}$$

$$4m - 2S = 3 \tag{63b}$$

$$3 \frac{a_1}{a_2} m + \left(24 - 5 \frac{a_1}{a_2}\right) S = 6 \tag{63c}$$

from which

$$S = \frac{24 - 9(a_1/a_2)}{96 - 14(a_1/a_2)} , \qquad m = \frac{84 - 15(a_1/a_2)}{96 - 14(a_1/a_2)} . \tag{64}$$

If we adjust the ratio a_1/a_2 to correspond to the Taylor value of .374 for $p_1/p(R_o)$, we obtain

$$a_1/a_2 = 1.252 \qquad \text{and} \tag{65a}$$

$$S = .163, \qquad m = .831, \tag{65b}$$

in very good agreement with the values of (40d).

It must be emphasized that due to the equations of motion, in the form

$$\rho(\vec{r}) \dot{u}_L(\vec{r}) + \nabla_L p(\vec{r}) = 0 , \tag{66}$$

relation (62) becomes an identity on insertion of (56) and (59). Thus we fail to be surprised when on numerical

computation of a_1 and a_2 from (57), (60):

$$a_1 = 1.101, \quad a_2 = .879, \quad a_1/a_2 = 1.252, \tag{67}$$

the ratio a_1/a_2 is precisely as obtained above. The important fact is that when the constants a_1, a_2 resulting from this identity for a uniform medium are carried over to the non-uniform case, they still appear to carry the major part of the thickness connection with them.

D. Other Corrections

We have seen that the snowplow model, with nominal corrections to the longitudinal flow, yields excellent agreement with the shock distortion of the (presumably correct) perturbation treatment, as well as a good estimate for the pattern of mass flow. Further improvement may be expected as the internal structure of the flow is taken further into account. Thus, since the integrated mass unit which we have been considering moves on a curve, there must be specific transverse effects taking place: the transverse acceleration may be imposed by transverse pressure gradients and may be reduced by a transverse mass flow.

The transverse pressure gradient is due to two related effects. First, even if the shock surface pressure were constant, the radial component of the transverse direction would extract a non-vanishing gradient from the unperturbed Taylor pattern. Second the shock surface pressure is not constant, and so there exists a transverse gradient which decays on passing into the interior. The mass conservation equation is also modified by the transverse flow, for not only is there a direct mass flow MU_T, but there is as well an indirect flow due to the change of the elementary mass cone in shape and orientation, aside from that of maintaining its unit shock surface, which has already been taken into account.

If one includes the aforementioned transverse effects

in a manner similar to that employed for the longitudinal corrections (gleaning most of the information from the original Taylor pattern) only the mean transverse velocity parameter b is required. The additional transverse force equation which is now available then enables S, m, and b to be determined. The value of b thus found, while always small, as in (40d), is quite sensitive to the precise formulation. It appears then that a somewhat haphazard patching up of the original model to account for these additional details of the hydrodynamical structure is not an effective way to proceed. Rather, one should set up a formalism in which the ideal snowplow approximation appears at an early stage, but which may then be systematically improved. Such an approach has been developed and will be presented at a later date.

E. A Simplified Approach

It is interesting to point out that one can describe a very simple procedure for obtaining good numerical values for the parameters S and m, at least in the case $\gamma = 1.4$, by choosing appropriate aspects of the snowplow picture. Although without great analytical justification, the method is readily generalized and may be of practical value. We reason as follows: Neglecting the relatively small surface distortion and transverse mass flow, the variation of the mass accumulation at a unit surface, due simply to the variation of ambient density, is

$$M(R,\theta) = \int_o^R \frac{r^2}{R^2} \rho_1(r,\theta)\, dr \ , \tag{68}$$

or inserting (4),

$$M = \frac{\rho_o R}{3} - \frac{\rho_o R^2}{4} \frac{1}{\lambda} \cos\theta \ , \tag{69}$$

whence $m = 3/4$.

If we now append the simple longitudinal force law (48) and its concomitant in (55a) for our special variation of ρ,

$$\nabla \cdot M\vec{U} = p_1/U \ , \qquad 5m - 23S = 0 \tag{70}$$

we obtain

$$S = \frac{15}{92} = .163. \tag{71}$$

F. Concluding Remarks

Our aim has been to investigate the adequacy of the "snowplow" type of model for analyzing blast propagation in non-uniform media. As a standard of comparison, the case of strong shock in a medium of $\gamma = 1.4$, and deviating only slightly from homogeneity, was solved numerically. For this special case, the snowplow with longitudinal corrections (Section C) and the empirically constructed model (Section E) reproduced exceedingly well those quantities which they were suitable for determining: shock surface position (and the surface values of hydrodynamical quantities obtainable therefrom) and total mass per unit surface. Since neither the strong shock boundary conditions nor the smallness of the inhomogeneity are necessary for the validity of the model or feasibility of the computations with it, it appears that such models may be valuable tools for the practical solution of shock hydrodynamical problems for intermediary shock strength and for large density and pressure inhomogeneities (with γ not required constant) as well.

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Appendix

Compilation of Coefficients for Equation (24)

Define $D \equiv f - g(y - h)^2$. Then

$$B_1 = \left[\frac{3}{\gamma} f + \frac{2y}{\gamma} f' + g(y - h)\left(\frac{3}{2} h + 2y h'\right)\right] D^{-1}$$

$$B_2 = f'/\gamma g(y-h)$$

$$B_3 = \left[\gamma f\left(\frac{3}{2} h + 2y h'\right) + (y-h)(3f+2yf')\right] D^{-1}$$

$$B_4 = \left[B_1 - 2y\left(\frac{g'}{g}\right)\right] (y - h)^{-1}$$

- - - - - - - - - - - -

$$A_{11} = \left[-f' - \frac{2\gamma f}{y} + \gamma g(y-h)\left(\frac{1}{2} - h'\right)\right] (\gamma D)^{-1}$$

$$A_{12} = -2f/yD$$

$$A_{13} = [2 - \gamma h' - 2\gamma h/y]\,(g/\gamma D)$$

$$A_{14} = f'(y - h)/\gamma D$$

- - - - - - - - - - - -

$$A_{21} = 0$$

$$A_{22} = \left(\frac{h}{y} - \frac{1}{2}\right)/(y-h)$$

$$A_{23} = -[\gamma y(y-h)]^{-1}$$

$$A_{24} = 0$$

- - - - - - - - - - - -

$$A_{31} = [\gamma f(\tfrac{1}{2} - h') - (y-h)\,(f' + \tfrac{2\gamma f}{y})]\; D^{-1}$$

$$A_{32} = -\,2\gamma f\,(y-h)/yD$$

$$A_{33} = \gamma(y-h)\,A_{13} - (g'/g)$$

$$A_{34} = (ff'/gD)$$

- - - - - - - - - - - -

$$A_{41} = [A_{11} + \frac{2}{y} + \frac{g'}{g}]\;(y-h)^{-1}$$

$$A_{42} = -\,2g(y-h)/yD$$

$$A_{43} = A_{13}(y-h)^{-1}$$

$$A_{44} = (f'/\gamma D) + (y-h)^{-1}$$

In the limit as $y \to 0$ the above go over to

$$\bar{B}_1 = \frac{3}{\gamma}\,, \quad \bar{B}_2 = \frac{\frac{5}{2} - \frac{1}{\gamma}}{\gamma-1}\,, \quad \bar{B}_3 = y[\frac{13}{2} - \frac{3}{\gamma}]\,, \quad \bar{B}_4 = \frac{-3(\gamma+1)}{y(\gamma-1)^2}$$

$$\bar{A}_{11} = -\,\frac{2}{y}\,, \quad \bar{A}_{12} = -\,\frac{2}{y}\,, \quad \bar{A}_{13} = 0\,, \quad \bar{A}_{14} = 0$$

$$\bar{A}_{21} = 0\,, \quad \bar{A}_{22} = \frac{1 - \gamma/2}{y(\gamma-1)}\,, \quad \bar{A}_{23} = \frac{-1}{y^2(\gamma-1)}\,, \quad \bar{A}_{24} = 0$$

$$\bar{A}_{31} = 1 - \frac{3\gamma}{2}\,, \quad \bar{A}_{32} = 2(1-\gamma)\,, \quad \bar{A}_{33} = \frac{-3}{y(\gamma-1)}\,, \quad \bar{A}_{34} = y[\frac{5}{2} - \frac{1}{\gamma}]$$

$$\bar{A}_{41} = \frac{3\gamma}{y^2(\gamma-1)^2}\,, \quad \bar{A}_{42} = 0\,, \quad \bar{A}_{43} = 0\,, \quad \bar{A}_{44} = \frac{\gamma}{y(\gamma-1)}$$

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